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OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

T. O. I.

NR-041-032

Technical Report No. 108

ON THE DYNAMIC BEHAVIOR OF PLASTIC-RIGID BEAMS UNDER TRANSVERSE LOAD

by

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PROVIDENCE, R. I.

April, 1954

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On the Dynamic Behavior of Plastic-Rigid Beams under Transverse Load[1]

by

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Abstract.

This paper extends previous work on the dynamic response of simply supported and clamped beams to transverse impact loading. Symonds,[1] [3], has given the solutions for these problems when the load-time relation is of a type described as "blast loading". In this paper solutions are obtained for the deformations caused by a general shape of the load-time curve. Numerical examples are computed for a symmetrically triangular load-time relation and compared with a rectangular load-time relation.

Introduction.

A recent paper written by Symonds, [1], treats the dynamical problems of simply supported and built-in beams, loaded with a concentrated force at the midpoint or a uniformly distributed load of magnitude such that large plastic deformations occur. It is there shown that two types of plastic deformations appear at high enough loads. There is either a moving hinge between two rotating rigid parts or one rotating rigid part and

1. The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract N7onr-35801 with Brown University.

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3. Numbers in square brackets refer to the bibliography at the end of the paper.

one finite plastic region, where the fully plastic moment is developed at all sections. The solutions of the second type of problems are restricted in [1] to load-time functions P(t) which are maximum at t = 0 and satisfy the inequality $Pt \leq \int_0^t P dt$. The aim of the present analysis is to give solutions for an arbitrary load-time relation for those problems where fully plastic regions appear. Those are the built-in beams with either a concentrated load at the midpoint or uniformly distributed load, and the simply supported beam with uniformly distributed load. Simply supported beams with a concentrated load at the midpoint are treated in full generality by Symonds, [1].

The same assumptions as in [1] and in two other papers [2] and [3] are used. These involve a plastic-rigid treatment; elastic deformations are assumed to be negligible. In [2] and [3] appear discussions about the conditions under which this analysis may give a reasonable answer.

A. The Built-in Beam with Concentrated Load

Those properties of the beam which enter into the problem are defined in Fig. 1. φ is the angle from the horizontal to the rigid part of the beam, ω and ω_0 are the angular velocities, and δ is the deflection of the midpoint of the beam. M_0 is the fully plastic moment and m is the mass per unit length. The force P(t) has the general form, shown in Fig. 2. The same dimensionless quantities as in the previous papers are used. They are

$$\mu = \frac{P\ell}{M_o} \qquad \Omega = \frac{m\ell^3\omega}{M_oT} \qquad \Phi = \frac{m\ell^3\varphi}{M_oT^2}$$

$$\eta = \frac{t}{T} \qquad \Omega_o = \frac{m\ell^3\omega_o}{M_oT} \qquad \Delta = \frac{m\ell^2\delta}{M_oT^2} .$$

T is an arbitrary reference time, e.g., the time during which a rectangular impulse acts.

At the start of the loading there is such a small force that the beam does not move. This is the first phase, which ends when the motion, described in Fig. 3 starts. With $\dot{\omega} = 0$ we obtain

$$\frac{1}{2} P\ell = 2M_o.$$

Hence the first phase ends when $\mu = \mu_I = 4$.

The second phase takes place for loads exceeding μ_I, see Fig. 3. The moment equilibrium equation about the built-in section yields

$$\frac{1}{2} P\ell = 2M_o + \frac{1}{3} m\ell^3\dot{\omega}$$

or

$$\Omega' = \frac{3}{2}\mu - 6. \tag{1}$$

Dot denotes d/dt and prime d/dη. Successive integrations give Ω and Φ. This phase ends when a region of fully plastic moment develops from the built-in end. This appears when R = 0,(Fig. 3).

$$R = \frac{1}{2} P - \frac{1}{2} m\ell^2\dot{\omega} = 0$$

or

$$\Omega' = \mu.$$

Substituting this into the above expression for Ω' determines $\mu_{II} = 12$. Hence the second phase applies for $4 \le \mu \le 12$.

For $\mu > \mu_{II}$ a configuration as in Fig. 1 appears. We must here distinguish between two cases, namely $\dot{\xi} \leq 0$ and $\dot{\xi} \geq 0$. If $\dot{\xi} \leq 0$, which we here call the third phase, the fully plastic region is increasing in length. The angular velocity of the element in the fully plastic region is described by ω_o. As there is no shear force acting within this region, and there are no exterior forces, the velocity and angular velocity must remain constant as long as the element remains within the plastic region. Hence ω_o is a function of x only. The element has the same angular velocity as the inner part of the beam had when the interface just passed it. In the Appendix of [2] the acceleration condition across a moving hinge is derived. In this case there is no difference between the angular velocity on each side of the moving interface and so the accelerations are continuous. Therefore the accelerations of the middle part of the beam are given by $(\xi\ell - x)\dot{\omega}$, Fig. 4. The moment equation about the midpoint of the beam and the equilibrium equation give

$$\left.\begin{aligned} 2M_o &= \frac{1}{6} m\xi^3\ell^3\dot{\omega} \\ \frac{1}{2} P &= \frac{1}{2} m\xi^2\ell^2\dot{\omega} \end{aligned}\right\}$$

or

$$\left.\begin{aligned} \xi^3\Omega' &= 12 \\ \xi^2\Omega' &= \mu \, . \end{aligned}\right\}$$

From these we have

$$\xi = \frac{12}{\mu}, \qquad \Omega' = \frac{\mu^3}{144}. \tag{2}$$

Two integrations give Ω and Φ.

Phase IV considers the case $\dot{\xi} \geq 0$. There is now a difference between the angular velocities on each side of the moving interface. This introduces, see [2], a jump in the acceleration across the moving interface of value $\dot{\xi}\ell[\omega - \omega_o(\xi)]$. The inner part of the beam therefore has the acceleration $(\xi\ell - x)\dot{\omega} + \dot{\xi}\ell[\omega - \omega_o(\xi)]$, Fig. 5. The moment equation about the midpoint of the beam and the equilibrium equation become

$$\left.\begin{aligned} 2M_o &= \frac{1}{2} m\xi^2\dot{\xi}\ell^3[\omega - \omega_o(\xi)] + \frac{1}{6} m\xi^3\ell^3\dot{\omega} \\ \frac{1}{2} P &= m\xi\dot{\xi}\ell^2[\omega - \omega_o(\xi)] + \frac{1}{2} m\xi^2\ell^2\dot{\omega} \end{aligned}\right\}$$

or

$$\left.\begin{aligned} 12 &= \xi^3\Omega' + 3\xi^2\xi'[\Omega - \Omega_o(\xi)] \\ \mu &= \xi^2\Omega' + 2\xi\xi'[\Omega - \Omega_o(\xi)]. \end{aligned}\right\}$$

From these we obtain

$$\Omega' = \frac{3\mu\xi - 24}{\xi^3}$$

$$\Omega = \Omega_o(\xi) + \frac{12 - \mu\xi}{\xi^2\xi'} .$$

Eliminating Ω by taking the derivative with respect to time of the second equation yields

$$\frac{\mu'\xi^2}{\xi'} + 2\mu\xi + \frac{12 - \mu\xi}{\xi'^2}\,\xi\xi'' = \xi^3\,\frac{d\Omega_o}{d\xi}\,\xi'$$

$$\frac{d}{d\eta}\left[\frac{\mu\xi^2}{\xi'} - \frac{12\xi}{\xi'} + 12\eta\right] = \xi^3\,\frac{d\Omega_o}{d\xi}\,\xi'$$

$$\mu = \frac{12}{\xi} - 12\eta \frac{\xi'}{\xi^2} + \frac{\xi'}{\xi^2} \left[\int_{\xi_o}^{\xi} \xi^3 \frac{d\Omega_o}{d\xi} d\xi + C_1 \right]$$

$$\int_{\eta_o}^{\eta} \mu \, d\eta = \frac{12\eta - C_1}{\xi} + \int_{\xi_o}^{\xi} \frac{1}{\xi^2} \left[\int_{\xi_o}^{\xi} \xi^3 \frac{d\Omega_o}{d\xi} d\xi \right] d\xi + C_2. \tag{3}$$

As Ω_o is a given function of ξ, the two integrations on the right hand side can be performed. On the left hand side is the given impulse. Hence this equation yields $\xi(\eta)$. $\Omega(\eta)$ is then determined by

$$\Omega = \Omega_o(\xi) + \frac{12 - \mu\xi}{\xi^2 \xi'}$$

or

$$\Omega = \Omega_o(\xi) + \frac{1}{\xi^3} \left[12\eta - C_1 - \int_{\xi_o}^{\xi} \xi^3 \frac{d\Omega_o}{d\xi} d\xi \right]. \tag{4}$$

The two constants of integrations C_1 and C_2 are determined by the values of ξ and Ω that apply at the beginning of phase IV. η_o and ξ_o denote the values of η and ξ at the beginning of phase IV. This phase ends when $\xi = 1$ or $\dot{\xi} = 0$.

The motion following phase IV for a decreasing load is governed by the equations given in phase II, with the new initial conditions. The problem ends when $\Omega = 0$. Note that during the last part of the problem we may have vanishing μ.

B. The Simply Supported and the Built-in Beam with Uniformly Distributed Load

These two problems are very similar and are treated simultaneously. The difference appears at the supported ends. When motion takes place the moment at the supported end is kM_o, where k = 0 for the simply supported beam and k = 1 for the built-in beam.

The uniformly distributed load has a total value P. During the first phase there is no motion. This phase ends when the motion of Fig. 6 starts. We obtain with $\dot{\omega} = 0$

$$(1 + k)M_o = \frac{1}{4} P\ell$$

or

$$\mu_I = 4(1 + k).$$

Hence phase I applies for $0 \leq \mu \leq 4(1 + k)$.

During the second phase the motion of Fig. 6 is applicable. The moment equation about the supported end yields

$$(1 + k)M_o + \frac{1}{3} m\ell^2\dot{\omega} = \frac{1}{4} P\ell$$

or

$$\Omega' = \frac{3}{4}\mu - 3(1 + k). \tag{5}$$

Successive integrations give Ω and Φ. When checking for the maximum moment we find in this case that a region of fully plastic moment starts from the midpoint. It begins to develop when the acceleration of the midpoint reaches the value $P/2\ell m$. Hence

$$\frac{P}{2m\ell} = \ell\dot{\omega}$$

or

$$\Omega' = \frac{1}{2}\mu.$$

Substituting this into the above expression for Ω' determines $\mu_{II} = 12(1 + k)$. Hence the second phase applies for $4(1 + k) \leq \mu \leq 12(1 + k)$.

For $\mu > \mu_{II}$ the motion of Fig. 7 takes place. In phase III we assume $\dot{\xi} \leq 0$. This implies that the acceleration is continuous across the moving interface. The acceleration condition at $x = \xi\ell$ and the moment equation about the supported end take the form

$$\left.\begin{aligned} \xi\ell\dot{\omega} &= \frac{P}{2m\ell} \\ (1 + k)M_o + \frac{1}{3}\xi^3\ell^3 m\dot{\omega} &= \frac{1}{4}\xi^2\ell P \end{aligned}\right\}$$

or

$$\left.\begin{aligned} \xi\Omega' &= \frac{1}{2}\mu \\ \xi^3\Omega' &= \frac{3}{4}\xi^2\dot{\mu} - 3(1 + k). \end{aligned}\right\}$$

From these we obtain

$$\xi = 2\sqrt{\frac{3(1 + k)}{\mu}}, \tag{6}$$

$$\Omega' = \frac{\mu\sqrt{\mu}}{4\sqrt{3(1 + k)}}. \tag{7}$$

Two integrations give Ω and Φ.

In phase IV we consider $\dot{\xi} \geq 0$. Compared to phase III there is a difference only in the acceleration condition across the moving interface. Because of the jump which has the value $\dot{\xi}\ell[\omega - \omega_o(\xi)]$, the acceleration condition across the interface and the moment equilibrium condition about the supported end take the form

$$\left.\begin{aligned} \xi \ell \dot{\omega} + \dot{\xi}\ell[\omega - \omega_o(\xi)] &= \frac{P}{2m\ell} \\ (1 + k)M_o + \frac{1}{3}\xi^3\ell^3 m\dot{\omega} &= \frac{1}{4}\xi^2 \ell P \end{aligned}\right\}$$

or

$$\left.\begin{aligned} \mu &= 2\xi\Omega' + 2\xi'[\Omega - \Omega_o(\xi)] \\ \mu\xi^2 &= \frac{4}{3}\xi^3\Omega' + 4(1 + k). \end{aligned}\right\}$$

Eliminating Ω in the same way as before we obtain

$$\frac{\mu'\xi^3}{\xi'} - \frac{\mu\xi^3}{\xi'^2}\xi'' + 3\mu\xi^2 + 12(1 + k) + \frac{12(1 + k)\xi\xi''}{\xi'^2} = 4\xi^3 \frac{d\Omega_o}{d\xi}\xi'$$

$$\frac{d}{d\eta}\left[\frac{\mu\xi^3}{\xi'} - \frac{12(1 + k)\xi}{\xi'} + 24(1 + k)\eta\right] = 4\xi^3 \frac{d\Omega_o}{d\xi}\xi'$$

$$\mu = \frac{12(1 + k)}{\xi^2} - \frac{24(1 + k)\eta\xi'}{\xi^3} + 4\frac{\xi'}{\xi^3}\left[\int_{\xi_o}^{\xi}\xi^3 \frac{d\Omega_o}{d\xi}d\xi + C_1\right]$$

$$\int_{\eta_o}^{\eta}\mu d\eta = \frac{12(1 + k)\eta - 2C_1}{\xi^2} + 4\int_{\xi_o}^{\xi}\frac{1}{\xi^3}\left[\int_{\xi_o}^{\xi}\xi^3 \frac{d\Omega_o}{d\xi}d\xi\right]d\xi + C_2. \qquad (8)$$

The constants of integration are determined in the same way as in the previous problem. We have

$$\Omega = \Omega_o(\xi) + \frac{12(1 + k) - \mu\xi^2}{4\xi^2\xi'}$$

or

$$\Omega = \Omega_o(\xi) + \frac{1}{\xi^3}\left[6(1 + k)\eta - C_1 - \int_{\xi_o}^{\xi}\xi^3 \frac{d\Omega_o}{d\xi}d\xi\right]. \qquad (9)$$

This phase ends when $\xi = 1$, or $\dot{\xi} = 0$.

The fifth and final phase has the same form and solution as phase II. New initial conditions apply. This phase ends when $\Omega = 0$. As before the load may vanish during this phase, or during the previous phase.

C. The Triangular Pulse for the Built-in Beam with a Concentrated Force at its Midpoint

In this section we compute the solutions of a typical problem which exemplifies the method described in the previous sections. We choose the built-in beam loaded with a concentrated load at its midpoint where the load-time relation is a symmetrical triangle, see Fig. 8:

$$P = P_m \frac{t}{T}, \quad 0 \leq t \leq T; \quad P = P_m\left(2 - \frac{t}{T}\right), \quad T \leq t \leq 2T; \quad P = 0, \; t \geq 2T$$

or

$$\mu = \mu_m \eta, \quad 0 \leq \eta \leq 1; \quad \mu = \mu_m(2 - \eta), \quad 1 \leq \eta \leq 2; \quad \mu = 0, \; \eta \geq 2.$$

We have to distinguish between three cases. The first consists of $4 \leq \mu_m \leq 12$, so that the fully plastic region never occurs. The second case consists of $12 \leq \mu_m \leq \mu_o$, where μ_o is determined by the condition that the force will not vanish during phase IV. The last case considers $\mu_m \geq \mu_o$.

In the first case only phases I and II take place. We have phase I when $0 \leq \eta \leq 4/\mu_m$. During phase II we have

$$\Omega' = \frac{3}{2} \mu_m \eta - 6 \tag{1a}$$

$$\Omega = \frac{3}{4} \mu_m \eta^2 - 6\eta + \frac{12}{\mu_m}$$

$$\Phi = \frac{1}{4} \mu_m \eta^3 - 3\eta^2 + \frac{12}{\mu_m} \eta - \frac{16}{\mu_m^2}.$$

This is valid for $\eta \leq 1$. We have at $\eta = 1$

$$\Omega = \frac{3}{4}\mu_m - 6 + \frac{12}{\mu_m}$$

$$\Phi = \frac{1}{4}\mu_m - 3 + \frac{12}{\mu_m} - \frac{16}{\mu_m^2} .$$

For $\eta \geq 1$ the force is given by $\mu = \mu_m(2 - \eta)$. Hence

$$\Omega' = \frac{3}{2}\mu_m(2 - \eta) - 6 \tag{1b}$$

$$\Omega = 3(\mu_m - 2)\eta - \frac{3}{4}\mu_m\eta^2 - \frac{3}{2}\mu_m + \frac{12}{\mu_m}$$

$$\Phi = \frac{3}{2}(\mu_m - 2)\eta^2 - \frac{1}{4}\mu_m\eta^3 - \left(\frac{3}{2}\mu_m - \frac{12}{\mu_m}\right)\eta + \frac{1}{2}\mu_m - \frac{16}{\mu_m^2} .$$

These expressions are valid until $\Omega = 0$ or $\eta = 2$. In the first case, when $\Omega = 0$, we obtain the following final values, when the motion ceased

$$\eta_f = 2 + \sqrt{2} - \frac{4(\sqrt{2} + 1)}{\mu_m}$$

$$\Phi_f^{(1)} = \frac{3}{2}\mu_m + \sqrt{2}\,\mu_m - 18 - 12\sqrt{2} + \frac{72}{\mu_m} + \frac{48\sqrt{2}}{\mu_m} - \frac{96}{\mu_m^2} - \frac{64\sqrt{2}}{\mu_m^2} .$$

This is valid for $\eta_f \leq 2$ i.e., $\mu_m \leq 4 + 2\sqrt{2}$. For $\mu_m \geq 4 + 2\sqrt{2}$ we obtain at $\eta = 2$

$$\Omega = \frac{3}{2}\mu_m - 12 + \frac{12}{\mu_m}$$

$$\Phi = \frac{3}{2}\mu_m - 12 + \frac{24}{\mu_m} - \frac{16}{\mu_m^2} .$$

For $\eta \geq 2$ we have

$$\Omega' = -6 \tag{1c}$$

$$\Omega = \frac{3}{2}\mu_m - 6\eta + \frac{12}{\mu_m}$$

$$\Phi = (\frac{3}{2}\mu_m + \frac{12}{\mu_m})\eta - 3\eta^2 - \frac{3}{2}\mu_m - \frac{16}{\mu_m^2} .$$

The motion ceases when $\Omega = 0$ or

$$\eta_f = \frac{1}{4}\mu_m + \frac{2}{\mu_m} .$$

The final angle becomes

$$\Phi_f^{(2)} = \frac{3}{16}\mu_m^2 - \frac{3}{2}\mu_m + 3 - \frac{4}{\mu_m^2} \qquad (4 + 2\sqrt{2} \leq \mu_m \leq 12).$$

For the second case, $12 \leq \mu_m \leq \mu_o$, phase I and II are the same as before, but in this case phase II ends when the load reaches the value μ_{II}, i.e., $\eta = 12/\mu_m$. For this value we have $\Omega_{II} = 48/\mu_m$ and $\Phi_{II} = 128/\mu_m^2$.

During phase III we have

$$\xi = \frac{12}{\mu_m\eta} \tag{2a}$$

$$\Omega' = \frac{\mu_m^3\eta^3}{144} \tag{2b}$$

$$\Omega = \frac{\mu_m^3\eta^4}{576} + \frac{12}{\mu_m}$$

$$\Phi = \frac{\mu_m^3\eta^5}{2880} + \frac{12\eta}{\mu_m} - \frac{102.4}{\mu_m^2} . \tag{2c}$$

This phase ends at $\eta = 1$. For this value we have

$$\xi_{III} = \frac{12}{\mu_m}, \quad \Omega_{III} = \frac{\mu_m^3}{576} + \frac{12}{\mu_m} \text{ and } \Phi_{III} = \frac{\mu_m^3}{2880} + \frac{12}{\mu_m} - \frac{102.4}{\mu_m^2}.$$

The angular velocity in the fully plastic region is $\Omega_o(x)$. It is determined by

$$\Omega_o(x) = \frac{\mu_m{}^3\eta^4}{576} + \frac{12}{\mu_m} \quad \text{where} \quad \eta = \frac{12\ell}{\mu_m x}.$$

Hence

$$\Omega_o(x) = \frac{36\ell^4}{\mu_m x^4} + \frac{12}{\mu_m}.$$

During phase IV we have

$$\int_1^\eta \mu_m(2 - \eta)d\eta = \frac{12\eta - C_1}{\xi} + \int_{12/\mu_m}^{\xi} \frac{1}{\xi^2}\left[\int_{12/\mu_m}^{\xi} \xi^3 \frac{d\Omega_o}{d\xi}\, d\xi\right] d\xi + C_2. \tag{3a}$$

We have

$$\frac{d\Omega_o(\xi)}{d\xi} = -\frac{144}{\mu_m \xi^5}.$$

Hence

$$\int_{12/\mu_m}^{\xi} \frac{1}{\xi^2}\left[\int_{12/\mu_m}^{\xi} \xi^3 \frac{d\Omega_o}{d\xi}\, d\xi\right]d\xi = -\frac{72}{\mu_m \xi^2} - \frac{\mu_m}{2} + \frac{12}{\xi}.$$

Moreover the initial conditions determine $C_1 = 12$ and $C_2 = 0$. Equation (3a) therefore yields

$$\left(\frac{12}{\mu_m \xi}\right)^2 - 2\eta\left(\frac{12}{\mu_m \xi}\right) - (2 - 4\eta + \eta^2) = 0$$

$$\frac{12}{\mu_m \xi} = \eta \pm \sqrt{2}\,(\eta - 1)$$

$$\xi = \frac{12}{\mu_m[\sqrt{2} - (\sqrt{2} - 1)\eta]}. \tag{3b}$$

The minus sign in front of the square root is chosen in order to ensure $\dot{\xi} \geq 0$. This phase ends when $\xi = 1$, i.e.,

$$\eta_{IV} = (\sqrt{2} + 1)(\sqrt{2} - \frac{12}{\mu_m}).$$

In order to have $\eta_{IV} \leq 2$ this case is restricted to

$$\mu_m \leq \mu_o = 12 + 6\sqrt{2}.$$

Inserting the above value of ξ in the expression for Ω, equation (4), yields

$$\Omega = \frac{\mu_m^3}{576}[(3\sqrt{2} + 1)\eta - 3\sqrt{2}][\sqrt{2} - (\sqrt{2} - 1)\eta]^3 + \frac{12}{\mu_m} \qquad (4a)$$

$$\Phi = \frac{\mu_m^3}{576}[2.8 + 0.8\sqrt{2} - 12\eta + 8(3 - \sqrt{2})\eta^2 - 4(7 - 4\sqrt{2})\eta^3$$
$$+ 6(3 - 2\sqrt{2})\eta^4 - \frac{1}{5}(23 - 16\sqrt{2})\eta^5] + \frac{12\eta}{\mu_m} - \frac{102.4}{\mu_m^2}. \qquad (4b)$$

At $\eta = \eta_{IV}$ we obtain

$$\Omega_{IV} = 12[2 + \sqrt{2} - 4(5 + 3\sqrt{2})\frac{1}{\mu_m}]$$

$$\Phi_{IV} = \frac{0.3 + 0.2\sqrt{2}}{144}\mu_m^3 - \frac{120 + 96\sqrt{2}}{\mu_m} + \frac{1049.6 + 806.4\sqrt{2}}{\mu_m^2}.$$

During phase V we have for $\eta_{IV} \leq \eta \leq 2$

$$\Omega' = \frac{3}{2}\mu_m(2 - \eta) - 6 \qquad (1d)$$

$$\Omega = 3(\mu_m - 2)\eta - \frac{3}{4}\mu_m\eta^2 - \frac{3}{2}\mu_m + \frac{12}{\mu_m}$$

$$\Phi = \frac{3}{2}(\mu_m - 2)\eta^2 - \frac{1}{4}\mu_m\eta^3 + (\frac{12}{\mu_m} - \frac{3}{2}\mu_m)\eta + \frac{0.3 + 0.2\sqrt{2}}{144}\mu_m^3$$
$$- (1 + \sqrt{2})\mu_m + 18 + 12\sqrt{2} - \frac{534.4 + 345.6\sqrt{2}}{\mu_m^2}.$$

At $\eta = 2$ we obtain

$$\Omega = \frac{3}{2}\mu_m - 12 + \frac{12}{\mu_m}$$

$$\Phi = \frac{0.3 + 0.2\sqrt{2}}{144}\mu_m^3 - \sqrt{2}\,\mu_m + 6 + 12\sqrt{2} + \frac{24}{\mu_m} - \frac{534.4 + 345.6\sqrt{2}}{\mu_m^2}.$$

During the second part of phase V we have

$$\Omega' = -6 \tag{1e}$$

$$\Omega = \frac{3}{2}\mu_m - 6\eta + \frac{12}{\mu_m}$$

$$\Phi = \left(\frac{3}{2}\mu_m + \frac{12}{\mu_m}\right)\eta - 3\eta^2 + \frac{0.3 + 0.2\sqrt{2}}{144}\mu_m^3$$

$$- (3 + \sqrt{2})\mu_m + 18 + 12\sqrt{2} - \frac{534.4 + 345.6\sqrt{2}}{\mu_m^2}.$$

The motion ends when $\Omega = 0$, i.e., $\eta_f = \frac{1}{4}\mu_m + \frac{2}{\mu_m}$. The final value of Φ is

$$\Phi_f^{(3)} = \frac{0.3 + 0.2\sqrt{2}}{144}\mu_m^3 + \frac{3}{16}\mu_m^2 - (3 + \sqrt{2})\mu_m + 21 + 12\sqrt{2}$$

$$- \frac{522.4 + 345.6\sqrt{2}}{\mu_m^2} \qquad (12 \leq \mu_m \leq 12 + 6\sqrt{2}).$$

In the third case $\mu_m \geq \mu_o = 12 + 6\sqrt{2}$. We have the same solutions as before until $\eta = 2$. At this time the force vanishes. We have at $\eta = 2$

$$\xi = \frac{6(2 + \sqrt{2})}{\mu_m}; \qquad \xi' = \frac{6(\sqrt{2} + 1)}{\mu_m}$$

$$\Omega = \frac{8\sqrt{2} - 11}{144}\mu_m^3 + \frac{12}{\mu_m}$$

$$\Phi = \frac{0.6\sqrt{2} - 0.7}{48}\mu_m^3 + \frac{24}{\mu_m} - \frac{102.4}{\mu_m^2}.$$

During the second part of phase IV $\mu = 0$. We obtain

$$0 = \frac{12\eta - C_1}{\xi} - \frac{144}{\mu_m} \int_{6(2+\sqrt{2})/\mu_m}^{\xi} \frac{1}{\xi^2} \left[\int_{6(2+\sqrt{2})/\mu_m}^{\xi} \frac{1}{\xi^2} d\xi \right] d\xi + C_2 \quad (3c)$$

or

$$\frac{12}{\mu_m \xi} = \eta + \sqrt{2} - 1 - \frac{C_1}{12} \pm \sqrt{(\eta + \sqrt{2} - 1 - \frac{C_1}{12})^2 - \frac{2C_2}{\mu_m}}.$$

The initial conditions at $\eta = 2$ determines

$$C_1 = 12(\sqrt{2} - 1), \qquad C_2 = \mu_m$$

and furthermore requires the minus sign in front of the square root. Hence

$$\frac{12}{\mu_m \xi} = \eta - \sqrt{\eta^2 - 2}$$

$$\xi = \frac{12}{\mu_m(\eta - \sqrt{\eta^2 - 2})}. \quad (3d)$$

Phase IV ends when $\xi = 1$. This gives $\eta_{IV} = \frac{\mu_m}{12} + \frac{6}{\mu_m}$. Inserting this value for ξ in the expression for Ω, equation (4), yields

$$\Omega = \frac{\mu_m^3}{576}[\eta - \sqrt{\eta^2 - 2}]^3[\eta + 3\sqrt{\eta^2 - 2}] + \frac{12}{\mu_m} \quad (4c)$$

$$\Phi = [-0.4\eta^5 + 2\eta^3 - 3\eta + 0.4(\eta^2 - 2)^{5/2} + 0.2\sqrt{2} + 0.7]\frac{\mu_m^3}{144} + \frac{12\eta}{\mu_m} - \frac{102.4}{\mu_m^2}. \quad (4d)$$

At the end of phase IV we have

$$\Omega_{IV} = \mu_m - \frac{24}{\mu_m}$$

$$\Phi_{IV} = \frac{(0.2\sqrt{2} + 0.7)\mu_m^3}{144} - \frac{\mu_m^2}{12} + 4 - \frac{73.6}{\mu_m^2} .$$

During phase V we have

$$\Omega' = -6 \tag{1f}$$

$$\Omega = \frac{3}{2}\mu_m - 6\eta + \frac{12}{\mu_m}$$

$$\Phi = (\frac{3}{2}\mu_m + \frac{12}{\mu_m})\eta - 3\eta^2 + \frac{(0.2\sqrt{2} + 0.7)\mu_m^3}{144} - \frac{3\mu_m^2}{16} - 3 - \frac{37.6}{\mu_m^2} .$$

The motion ends when $\Omega = 0$. This gives

$$\eta_f = \frac{1}{4}\mu_m + \frac{2}{\mu_m}$$

$$\Phi_f^{(4)} = \frac{(0.7 + 0.2\sqrt{2})\mu_m^3}{144} - \frac{25.6}{\mu_m^2} \qquad (\mu_m \geq 12 + 6\sqrt{2}).$$

Φ_f as a function of μ_m is plotted in Fig. 10.

We are also interested in the final deflection of the midpoint of the beam. The beam is straight for the part $0 \leq x \leq \bar{\xi}\ell$, where $\bar{\xi}$ is the minimum value of ξ during the motion: $\bar{\xi} = 12/\mu_m$. The part $\bar{\xi}\ell \leq x \leq \ell$ is curved due to the finite plastic region. Consider an element in this part of the beam. It becomes a member of the finite plastic region at time $\eta_1 = \frac{12\ell}{\mu_m x}$; (see equation (2a)). It remains in the finite plastic region until

$$\eta_2 = (\sqrt{2} + 1)(\sqrt{2} - \frac{12\ell}{\mu_m x}), \quad \eta_2 \leq 2; \quad \text{(see equation (3b))}$$

or

$$\eta_2 = \frac{\mu_m x}{12\ell} + \frac{6\ell}{\mu_m x}, \quad \eta_2 \geq 2; \quad \text{(see equation (3d))}.$$

During the time in the finite plastic region this element has the constant angular velocity $\Omega_o = \frac{36\ell^4}{\mu_m x^4} + \frac{12}{\mu_m}$. Hence its rotation during the time in the finite plastic region is $\Phi_o = (\frac{36\ell^4}{\mu_m x^4} + \frac{12}{\mu_m})(\eta_2 - \eta_1)$. During the remaining time of the motion the element is rigidly connected to the center part of the beam. At time η_1 the center part of the beam has the slope

$$\Phi_1 = \frac{86.4\ell^5}{\mu_m^2 x^5} + \frac{144\ell}{\mu_m^2 x} - \frac{102.4}{\mu_m^2}; \quad \text{(see equation (2d))}.$$

At time η_2 the center of the beam has the slope

$$\Phi_2^{(1)} = \frac{(0.3 + 0.2\sqrt{2})\mu_m^3}{144} - \frac{(144 + 108\sqrt{2})\ell^4}{\mu_m x^4} + \frac{(1296 + 950.4\sqrt{2})\ell^5}{\mu_m^2 x^5}$$

$$+ \frac{24 + 12\sqrt{2}}{\mu_m} - \frac{(144 + 144\sqrt{2})\ell}{\mu_m^2 x} - \frac{102.4}{\mu_m^2}$$

$$\eta_2 \leq 2; \quad \text{(see equation (4b))}.$$

$$\Phi_2^{(2)} = \frac{(0.7 + 0.2\sqrt{2})\mu_m^3}{144} - \frac{\mu_m^2 \ell}{12x} + \frac{3\ell^3}{x^3} - \frac{43.2\ell^5}{\mu_m^2 x^5}$$

$$+ \frac{x}{\ell} + \frac{72\ell}{\mu_m^2 x} - \frac{102.4}{\mu_m^2} \quad \eta_2 \geq 2; \quad \text{(see equation (4d))}.$$

Hence the final angle Φ_{of} in the part $\bar{\xi}\ell \le x \le \ell$ is given by

$$\Phi_{of}^{(1)} = \Phi_f^{(3)} - \Phi_2^{(1)} + \Phi_1 + \Phi_o, \quad \text{elements for which } \eta_2 \le 2$$

$$\Phi_{of}^{(2)} = \Phi_f^{(4)} - \Phi_2^{(2)} + \Phi_1 + \Phi_o, \quad \text{elements for which } \eta_2 \ge 2$$

We therefore have

$$\Delta_f^{(1)} = \Phi_f^{(1)} \quad \text{and} \quad \Delta_f^{(2)} = \Phi_f^{(2)} \qquad 4 \le \mu_m \le 12$$

$$\Delta_f^{(3)} = \Phi_f^{(3)}\bar{\xi} + \int_{\bar{\xi}}^{1} \Phi_{of}^{(1)} \, d\left(\frac{x}{\ell}\right) \qquad 12 \le \mu_m \le 12 + 6\sqrt{2}$$

$$\Delta_f^{(4)} = \Phi_f^{(4)}\bar{\xi} + \int_{\bar{\xi}}^{\xi_2} \Phi_{of}^{(1)} \, d\left(\frac{x}{\ell}\right) + \int_{\xi_2}^{1} \Phi_{of}^{(2)} \, d\left(\frac{x}{\ell}\right) \qquad \mu_m \ge 12 + 6\sqrt{2}$$

where

$$\xi_2 = \frac{12 + 6\sqrt{2}}{\mu_m}$$

Δ_f as a function of μ_m is plotted in Fig. 11.

We compare the above solutions with the solutions for a rectangular pulse defined by the values (Fig. 9)

$$P = P_m, \; 0 \le t \le T; \quad P = 0, \; t \ge T$$

or

$$\mu = \mu_m, \; 0 \le \eta \le 1; \quad \mu = 0, \; \eta \ge 1.$$

This problem is solved in [1]. The solutions are

$$\Phi_f = \frac{3}{16}\mu_m^2 - \frac{3}{4}\mu_m \qquad (4 \le \mu_m \le 12)$$

$$\Phi_f = \frac{\mu_m^3}{96} \qquad (\mu_m \ge 12)$$

$$\Delta_f = \frac{3}{16}\mu_m^2 - \frac{3}{4}\mu_m \qquad (4 \le \mu_m \le 12)$$

$$\Delta_f = \frac{\mu_m^2}{24}\left(3 + 2\,\ln\frac{\mu_m}{12}\right) \qquad (\mu_m \geq 12).$$

The final deformations for the rectangular pulse are plotted in Figs. 10 and 11.

The solutions have also been computed in a similar manner for the beams discussed in Section B for the symmetrically triangular load-time relation and the rectangular load-time relation (Figs. 12 - 15).

D. Summary

Equations (1) - (9) determine the motion of the beam for the general load-time relation of Fig. 2. In particular the final deformations have been computed for the triangular load-time relation of Fig. 8 and the results are shown in Figs. 10 - 15. In these figures the final deformations are also shown for a rectangular pulse, (Fig. 9), having the same total impulse as the triangular pulse. A comparison between these two pulses shows that the final deformations of the triangular pulse are smaller than those of the rectangular pulse and that the difference is 15% - 35% of the values of the rectangular pulse when the maximum load exceeds twice the static collapse load. Similar comparisons with the rectangular pulse have previously been made, see [4], and they show differences between the final deformations that are of the same order of magnitude as found in this paper. The conclusions of [4] can therefore be expected to be valid for the problems discussed in this paper.

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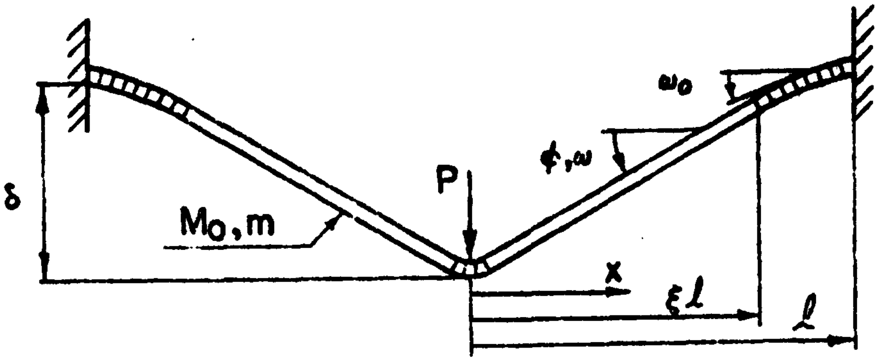


Fig. 1. The plastic deformations in the built in beam with a concentrated load at its midpoint

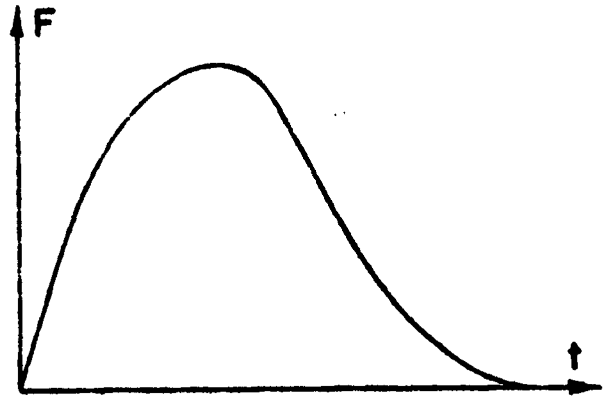
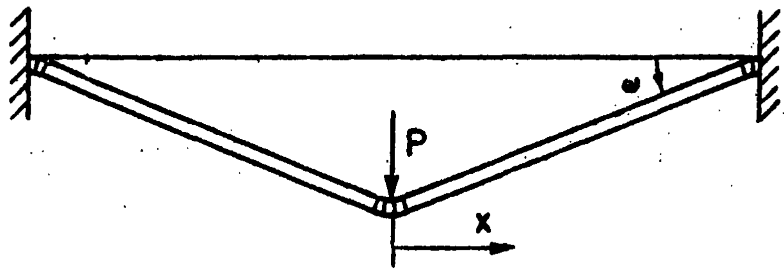


Fig. 2. The load-time relation



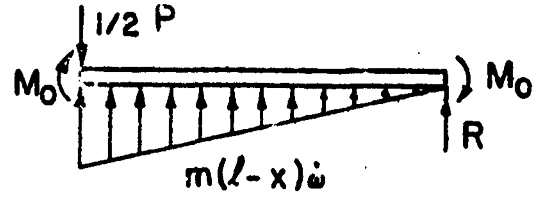


Fig. 3. The external and dynamic forces during phase II

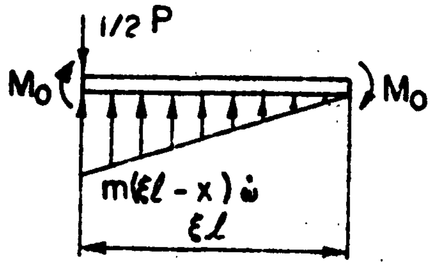


Fig. 4. The external and dynamic forces during phase III

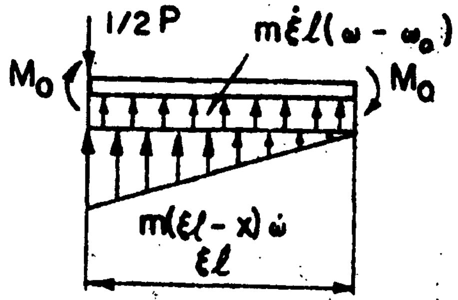


Fig. 5. The external and dynamic forces during phase IV

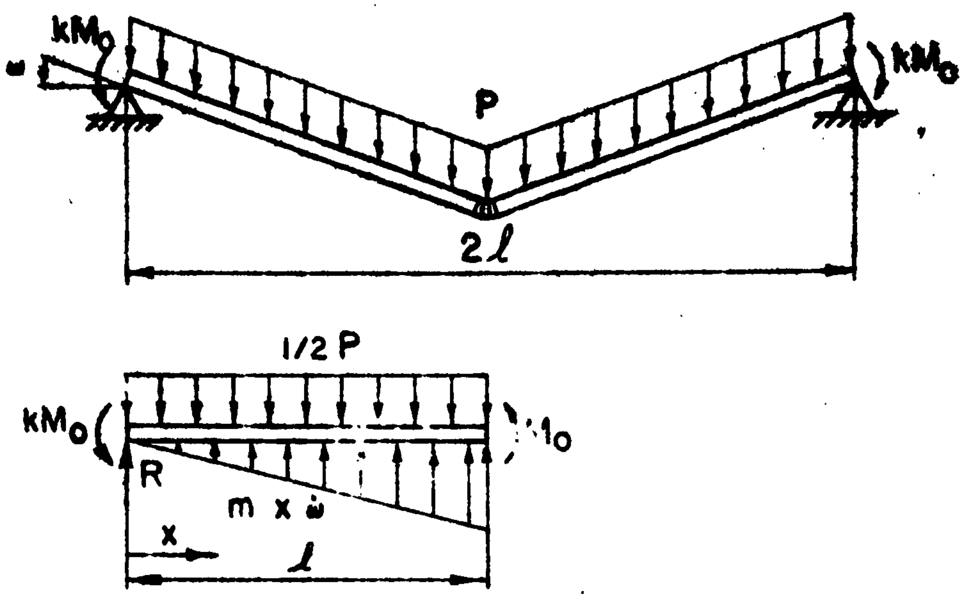


Fig. 6. The external and dynamic forces during phase II for the beam with uniformly distributed load

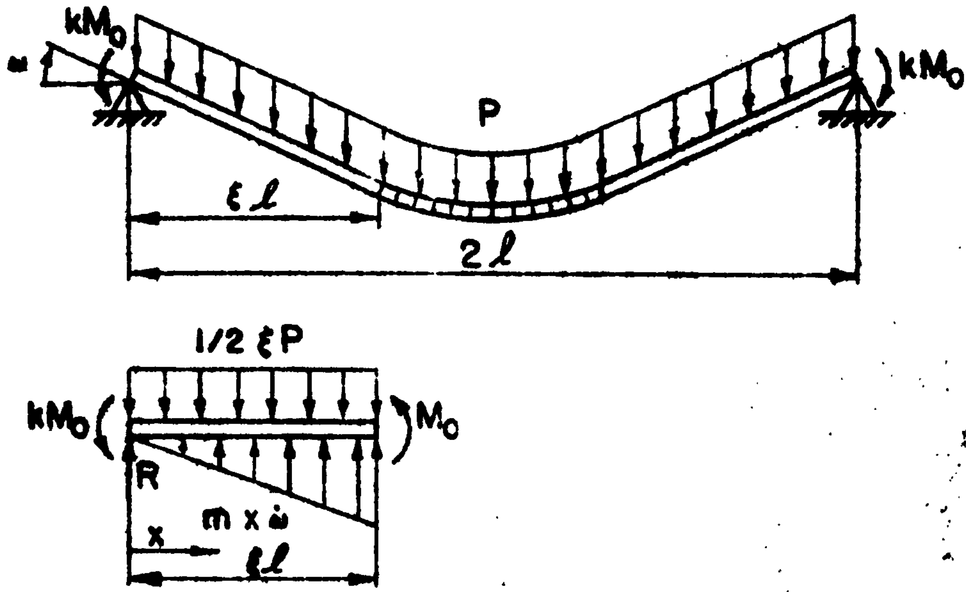


Fig. 7. The external and dynamic forces during phase III and IV for the beam with uniformly distributed load

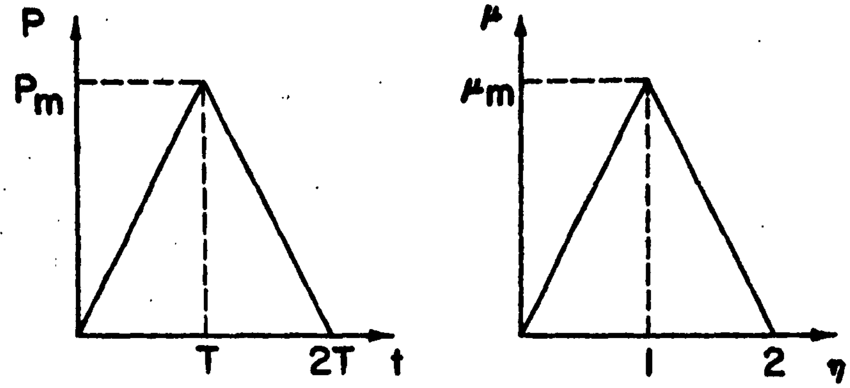


Fig. 8. The triangular load-time relation

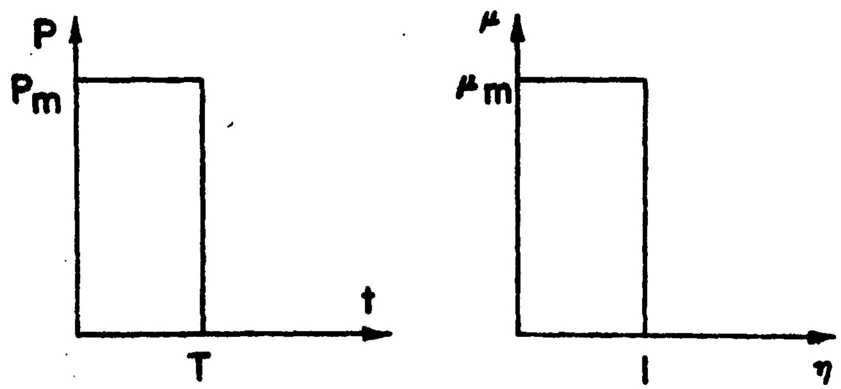


Fig. 9. The rectangular load-time relation

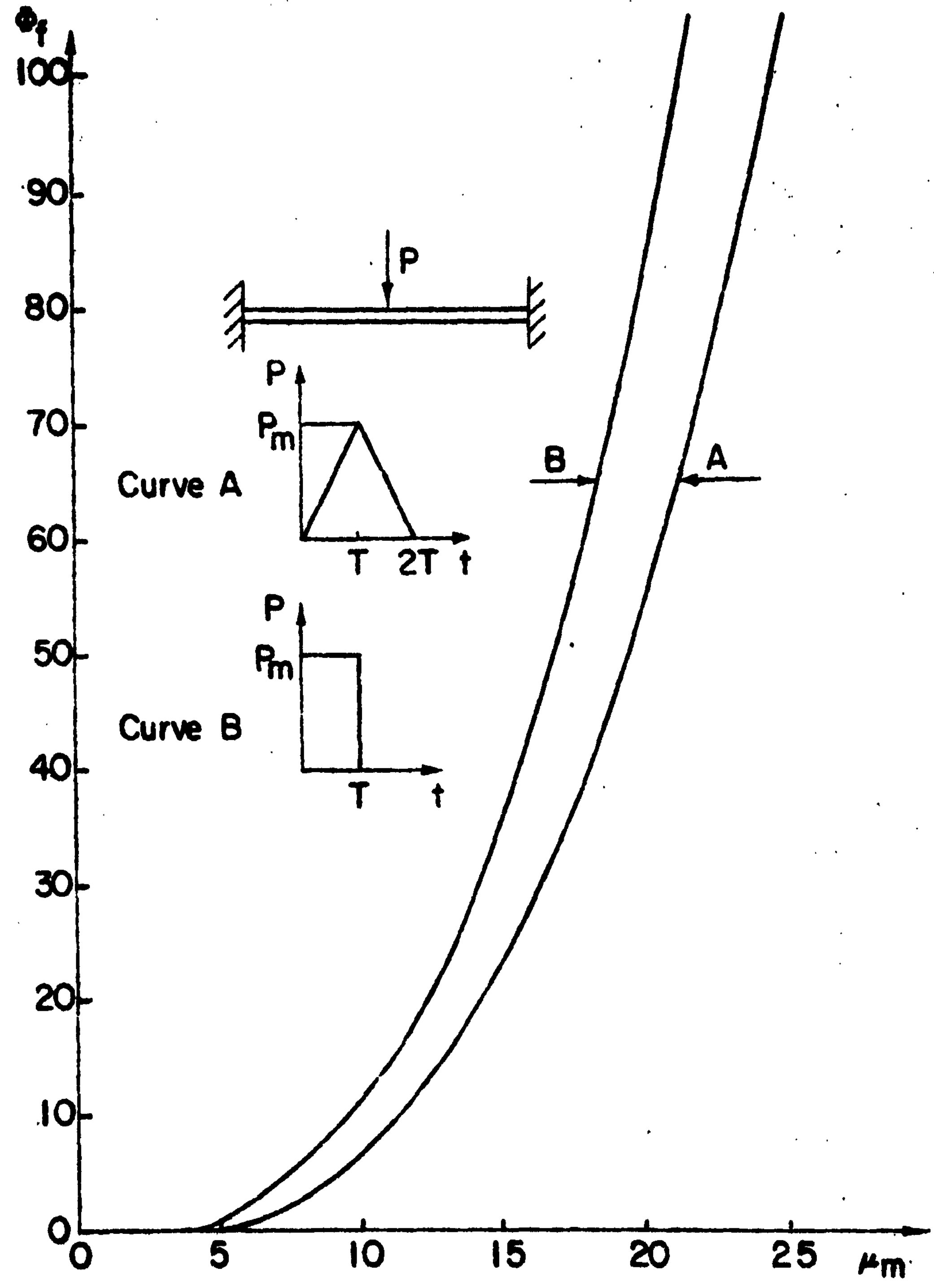


Fig. 10. The final slope of the non-curved part of the built-in beam with a concentrated load at its midpoint

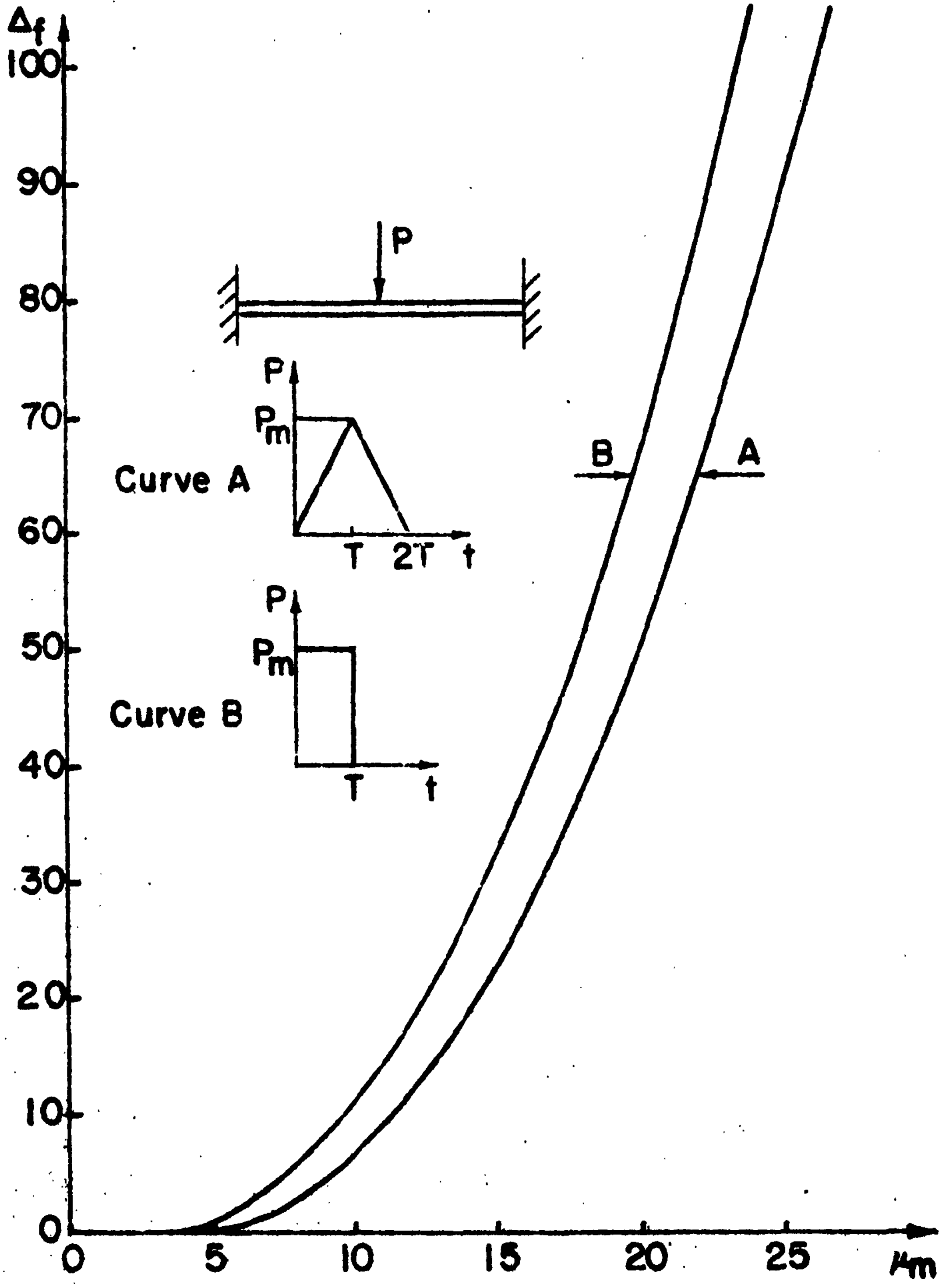


Fig. 11. The final deflection of the midpoint of the built-in beam with a concentrated load at its midpoint

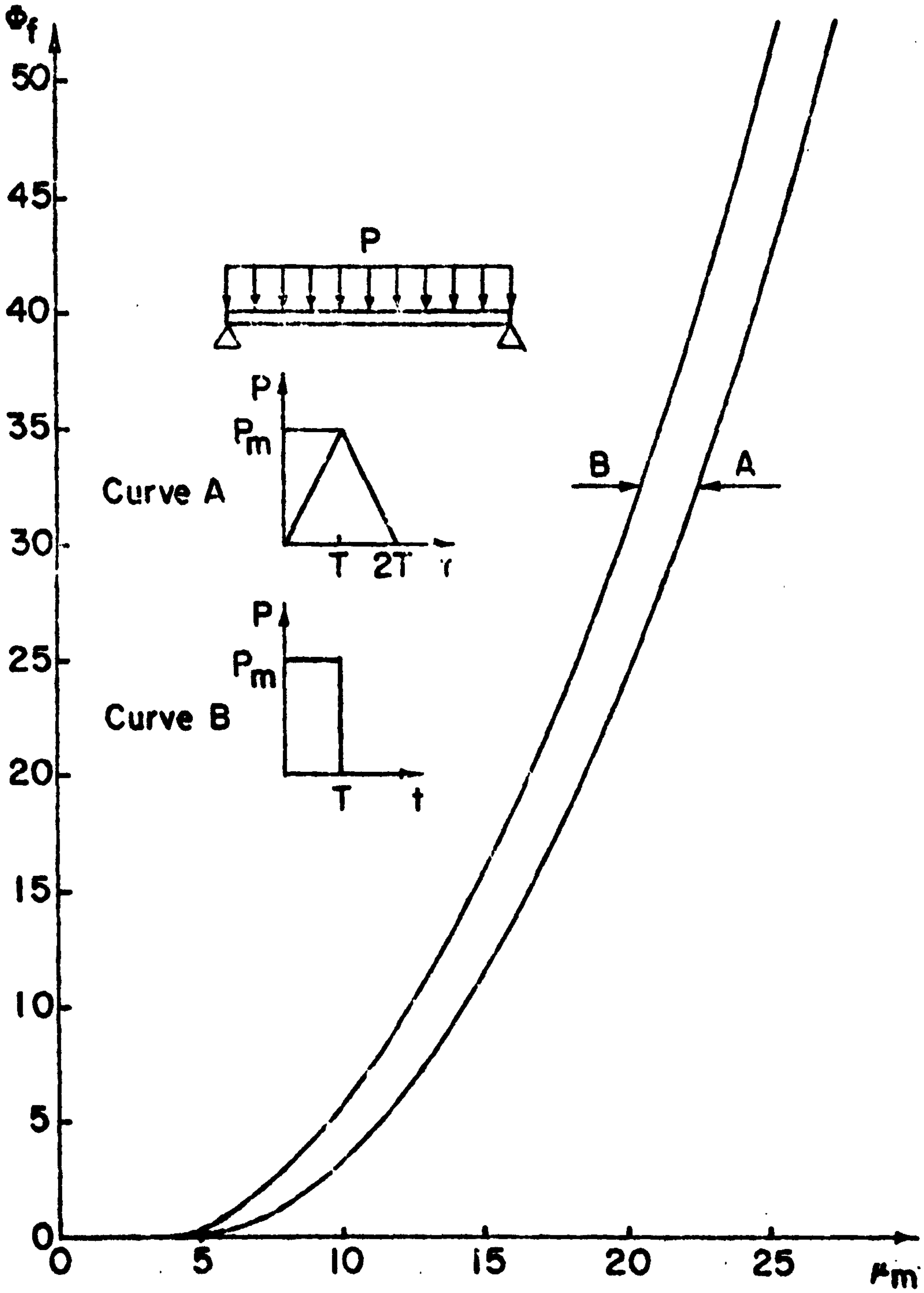


Fig. 12. The final slope of the non-curved part of the simply supported beam with uniformly distributed load

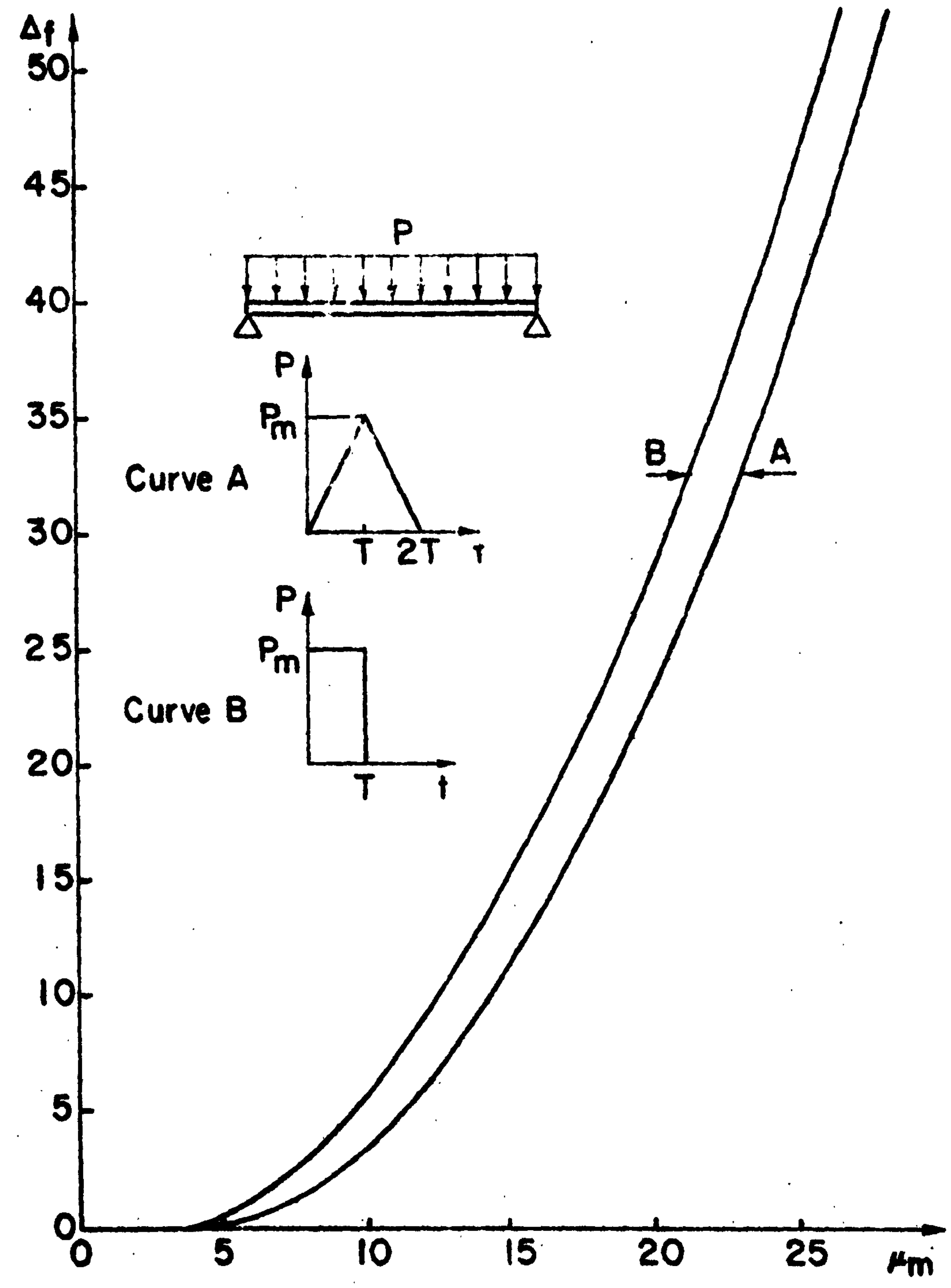


Fig. 13. The final deflection of the midpoint of the simply supported beam with uniformly distributed load

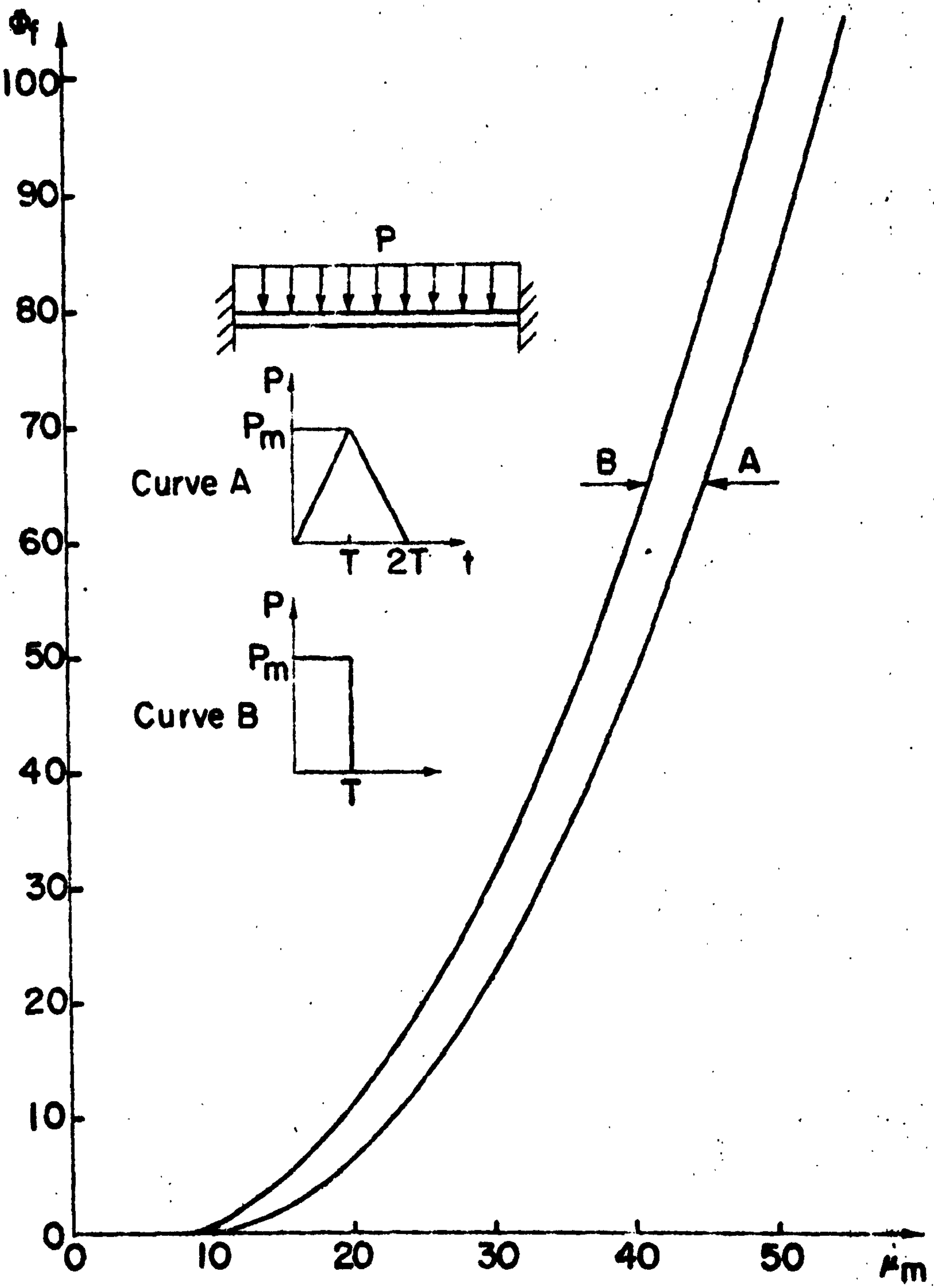


Fig. 14. The final slope of the non-curved part of the built-in beam with uniformly distributed load

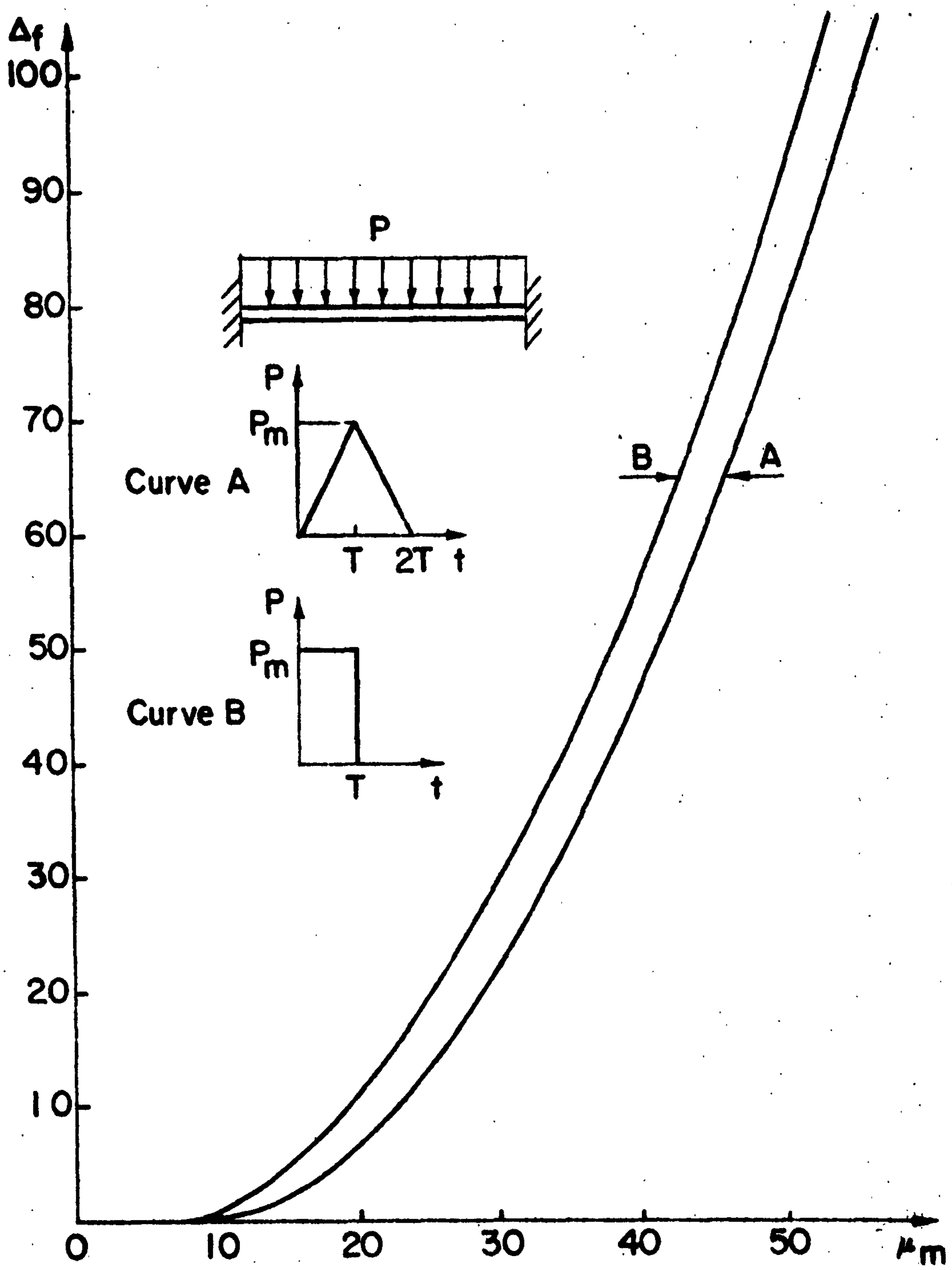


Fig. 15. The final deflection of the midpoint of the built in beam with uniformly distributed load